Solution of coupled nonlinear Schrödinger equations in a focusing-defocusing medium by modified perturbation theory

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Abstract—Interaction is considered of bright solitons of different orders and two different wavelengths propagating in a medium focusing for one wavelength and defocusing for the other. The system of nonlinear Schrödinger equations is solved by means of perturbation theory. The application of an additional postulate to adjust both widths of the solitons and to modify the amplitude by a factor determined by the overlap integral greatly improves the accuracy of the description. Good accuracy of description is confirmed by numerical calculations.

In nonlinear optics, the coupled nonlinear Schrödinger equations have been for many years the main tool for studying the interactions of solitons with each other and with the medium through which they pass [1–6]. In this paper, we consider a nonlinear medium focusing for a wave at one frequency and defocusing for another and the description of interaction between two such waves.

Consider two beams $U_{Pos}(x,z)$ and $U_{Neg}(x,z)$ interacting with a nonlinear medium of nonlinearity:

$$\varepsilon_2 = \alpha_P |U_{Pos}|^2 - \alpha_N |U_{Neg}|^2, \qquad (1)$$

for $\alpha_P > \alpha_N \geq 0$. The Nonlinear Schrödinger Equations (NSE) describing the propagation of beams have the form:

$$\begin{aligned} i\beta_P \frac{\partial U_{Pos}}{\partial z} + \frac{1}{2}\frac{\partial^2 U_{Pos}}{\partial x^2} + \left(\alpha_P |U_{Pos}|^2 - \alpha_N |U_{Neg}|^2\right) U_{Pos} = 0, \\ i\beta_N \frac{\partial U_{Neg}}{\partial z} + \frac{1}{2}\frac{\partial^2 U_{Neg}}{\partial x^2} + \left(\alpha_N |U_{Pos}|^2 - \alpha_N |U_{Neg}|^2\right) U_{Neg} = 0. \end{aligned} \qquad (2)$$

This case was considered by many authors [4, 6–10] and discussed using different attitudes – analytical [1, 4, 7, 9], numerical [10] or variational [6, 8].

But the simplest solution of Eqs. (2) describes the case of vanishing field U_{Neg}. Normalizing the wave function U_{Pos} together with the coordinates (x, z) gives NSE in its fundamental form:

$$i\frac{\partial \Psi_n}{\partial z} + \frac{1}{2}\frac{\partial^2 \Psi_n}{\partial x^2} + |\Psi_n|^2 \Psi_n = 0. \qquad (3)$$

The series of solutions of Eq. (3) can be obtained via Inverse Scattering Transform (IST) [4]. They represent solitons of different orders $n = 1, 2, \ldots$ The function Ψ_n is a quotient of two complex combinations of n terms of the form $\exp\left(-2\eta_l (x - x_{0l}) + 2i\eta_l^2 (z - z_{0l})\right)$ with arbitrary real η_l, x_{0l}, and z_{0l} ($l = 1,..n$). The coefficients η_l determine the widths and heights of its individual components.

A system of equations like Eqs. (2) for $\beta_P = \beta_N = 1$ and focusing nonlinearity for both beams $\alpha_P = 1$, $\alpha_N = -1$ has an analytical solution known as Manakov solitons [4, 10–14]. Based on Manakov's solution, an analogous solution of the system [Eqs. (2)] for focusing-defocusing nonlinearity has been discussed in [4]. Both fields U_{Pos} and U_{Neg} are described by solitons of the same order, but unfortunately this solution exists only for the special case where $\alpha_P = \alpha_N = 1$ and $\beta_P = \beta_N = 1$.

The considered system of Eqs. (2) generalizes this case. To obtain Manakov-type solution we should introduce additional amplitude factors γ_{Pos} and γ_{Neg}:

$$\begin{aligned} U_{Pos} = \gamma_{Pos} \Psi_n, \\ U_{Neg} = \gamma_{Neg} \Psi_n. \end{aligned} \qquad (4)$$

Assume the real γ_{Pos} and γ_{Neg} (their phases are included into initial phase factors of Ψ_n). Substituting Eqs. (4) into Eqs. (2) we can prove that Eq. (3) is satisfied only for:

$$\begin{aligned} \beta_P = \beta_N = 1, \\ \alpha_P \gamma_{pos}^2 - \alpha_N \gamma_{Neg}^2 = 1. \end{aligned} \qquad (5)$$

The requirement (4) implies that solitons of the both fields U_{Pos} and U_{Neg} have not only the same order n, but the centra x_{0l} of all their corresponding components for $l_{Pos} = l_{Neg} = 1,..\, n$ have the same positions. In addition, for any pair of component indices l_1 and l_2 the initial phase differences $\eta_{l1}^2 z_{0l1} - \eta_{l2}^2 z_{0l2}$ are the same for both fields.

In this paper we solve the system of Eqs. (2) applying perturbation theory. This attitude is frequently applied to describe various effects connected with interaction between solitons or with the medium [1, 5]. Although the U_{Neg} field will not always be small relative to U_{Pos}, let us treat it initially as a small perturbation. Expressing both fields as series with respect to a small constant quantity o of order of magnitude U_{Neg} we have:

$$\begin{aligned} U_{Pos} = U_P^{(0)} + oU_P^{(1)} + o^2 U_P^{(2)} + \ldots, \\ U_{Neg} = oU_N^{(1)} + \ldots. \end{aligned} \qquad (6)$$

The unperturbed field $U_P^{(0)}$ satisfies the first equation of the system (2) with vanishing U_{Neg}. Its solution describes a soliton of arbitrary order n:

$$U_P^{(0)}(x,z) = \gamma_P^{(0)} \Psi_n(x,z), \qquad (7)$$

with $\gamma_P^{(0)}$ denoting the same symbol as in Eqs. (4) and (5), but written for vanishing U_{Neg}. For $n = 1$ this solution gives:

$$U_P^{(0)} = \gamma_P^{(0)}\Psi_1 = A_P e^{i(k_P z+\varphi_{0P})} Sech(2\eta_P x), \tag{8}$$

with $k_P = 2\eta_P^2/\beta_P$ and $A_P = 2\eta_P/\sqrt{\alpha_P} = 2\eta_P\gamma_P^{(0)}$. The initial phase φ_{0P} is arbitrary.

By substituting the expansion (6) into the system (2), we prove that the first-order correction vanishes $U_P^{(1)} \equiv 0$ while $U_N^{(1)}$ satisfies the equation:

$$i\beta_N \frac{\partial U_N^{(1)}}{\partial z} + \frac{1}{2}\frac{\partial^2 U_N^{(1)}}{\partial x^2} + \alpha_N |U_P^{(0)}|^2 U_N^{(1)} = 0. \tag{9}$$

The equation for the *Neg* field $U_N^{(1)}$ is linear. Its solution gives a function with a profile like $U_P^{(0)}$, but it may have a different propagating term:

$$U_N^{(1)}(x,z) = a_N e^{i\Delta k z} U_P^{(0)}(x,z), \tag{10}$$

Nevertheless, substituting (10) into Eq. (9), we prove that it is satisfied only for two cases: 1) $\beta_N = \beta_P$, $\Delta k = 0$ or 2) $U_P^{(0)} = \Psi_1$ and $\Delta k = 2\eta_P^2(1/\beta_N - 1/\beta_P)$. Of particular interest is the second case, in which two solitons U_{Pos} and U_{Neg} can correspond to two different wavelengths.

But one can also consider solutions of Eq. (9), where instead of a true distribution of the refractive index derived from the nonlinearity $\alpha_N|U_P^{(0)}|^2$ we substitute a distribution that is close, but slightly different. Assume that this slightly different distribution is given by:

$$\left|U_P^{(0)}(x,z)\right|^2 \approx U_d^2 = A_d^2 Sech^2(2\eta_d x), \tag{11}$$

The parameter η_d can slightly differ from one of the numbers $\eta_{P1}...\eta_{Pn}$, defining *n-th* order soliton (7), but the relation between A_d and η_d are assumed the same as the first-order soliton: $A_d = 2\eta_d/\sqrt{\alpha_P}$. Moreover, to minimize deviation from the *z*-axis during propagation, assume an approximately symmetrical shape of the soliton $|U_P^{(0)}(x,z)|$ in the whole range of propagation.

The approximation (11) gives the solution of Eq. (9):

$$U_N^{(1)} = \gamma_N^{(1)}\sqrt{\alpha_P}\, U_d = A_N e^{ik_N z} Sech(2\eta_d x), \tag{12}$$

with $k_N = 2\eta_{Pd}^2/\beta_N$ but arbitrary amplitude $\gamma_N^{(1)}$ (or A_N) of U_{Neg} field (the multiplier before U_d is for consistency with designations in expressions (4)).

The first non-vanishing correction $U_P^{(2)}$ for U_{Pos} field satisfies a much more complicated equation:

$$i\beta_P \frac{\partial U_P^{(2)}}{\partial z} + \frac{1}{2}\frac{\partial^2 U_P^{(2)}}{\partial x^2} + \alpha_{Pos}\left(2|U_P^{(0)}|^2 U_P^{(2)} + \left(U_P^{(0)}\right)^2 U_P^{*(2)}\right) = \alpha_{Neg}\, |U_{Neg}^{(1)}|^2\, U_P^{(2)}. \tag{13}$$

Nevertheless, assuming $U_N^{(1)}$ of the fom (12) and:

$$U_P^{(2)} = \gamma_P^{(2)}\sqrt{\alpha_P}\, U_P^{(0)} \tag{14}$$

we are able to find the solution of Eq. (13) for real $\gamma_N^{(1)}$ and $\gamma_P^{(2)}$. Substituting (12) and (14) into Eq. (13) gives the condition for the existence of this solution:

$$\gamma_P^{(2)} = \alpha_{Neg}\frac{\left(\gamma_N^{(1)}\right)^2}{2\alpha_{Pos}\gamma_P^{(0)}}. \tag{15}$$

The obtained perturbed solutions of Eqs. (7)–(15) contain at least one case of different propagation constants $\beta_N \neq \beta_P$, but also cover the case of Manakov-type solution, Eqs. (4)–(5). To compare results following from these two attitudes, we should assume equal β_N and β_P, as in Eq. (5). Now we can see that treating $\gamma_{Neg} \approx o\gamma_N^{(1)}$ as a small quantity we have:

$$\gamma_{Pos} = \sqrt{\frac{1+\alpha_N\gamma_{Neg}^2}{\alpha_P}} \approx \frac{1}{\sqrt{\alpha_P}} + o^2\frac{\alpha_N\left(\gamma_N^{(1)}\right)^2}{2\sqrt{\alpha_P}} = \gamma_P^{(0)} + o^2\gamma_P^{(2)}. \tag{16}$$

Since the obtained second-order correction in expansion (16) is identical as (15), the conclusions from the exact Manakov-type solution (4)-(5) and the perturbation theory (7)–(15) are the same for the same choice of parameters. But the perturbation theory gives more possibilities to change the parameters of the interacting solitons.

By now the amplitude term $\gamma_N^{(1)}$ is arbitrary, because Eq. (9) is linear. To establish this term let us define the overlap integral Q calculated using the initial field shapes:

$$Q = \frac{\int_{-\infty}^{\infty}|U_P(x,0)|\cdot|U_N(x,0)|\,dx}{\sqrt{\int_{-\infty}^{\infty}|U_P(x,0)|^2\,dx\cdot\int_{-\infty}^{\infty}|U_N(x,0)|^2\,dx}}. \tag{17}$$

Of course, $0 \leq Q \leq 1$. During propagation the *Neg* soliton and central peak of the *Pos* soliton equal their widths. The amplitude of the U_{Neg} field will decrease or increase, depending on the relation between η_N and η_P. But its power always decreases to Q^2 of the initial power (U_{Neg} field is much smaller than U_{Pos}, so the power carried by *Pos* soliton hardly changes). This gives the rule enabling us to determine the amplitude of U_{Neg} field:

$$U_Q = \lim_{z\to\infty}|U_{Neg}(0,z)| = Q\cdot\sqrt{\frac{\eta_d}{\eta_N}}\cdot|U_{Neg}(0,0)|. \tag{18}$$

Of course, both solitons should retain their energy, as proved in [9]. But in the process of adjusting their widths some of the waves quickly escape outside the region of interaction. Thus, the reduction in power applies only to the fields that remain in the system still interacting with each other.

To check the obtained results numerically we assumed $\alpha_P = 1$, $\alpha_N = 0.25$, $\beta_P = 1$ and $\beta_N = 0.8$. As the input U_{Pos} beam we took a third-order soliton with the central peak corresponding to $\eta_P = 3$ at *z*-axis ($x_{P10} = 0$) and two side peaks approximately twice higher than the central peak. U_{Neg} beam at the input contained the first-order soliton defined by parameter $\eta_N = 10$ with amplitude 2/3 height of the central *Pos* peak. The other initial parameters have been adjusted to obtain the side peaks clearly separated from the central maximum. For these values in the input plane on the x-axis we have $\alpha_P |U_{Pos}|^2 \approx 0.7\alpha_N |U_{Neg}|^2$, which means that at the beginning of propagation the U_{Neg} beam reduces the nonlinear susceptibility over 3 times, so the interaction of solitons cannot be considered weak.

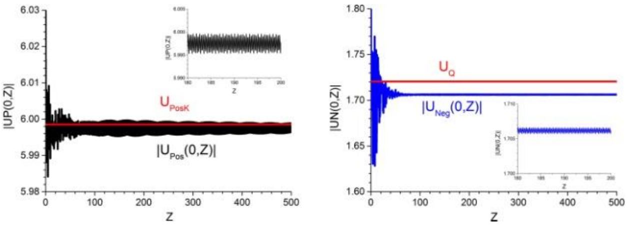

Fig. 1. Central heights of U_{Pos} and U_{Neg} fields during propagation over the distance z_K=500. $U_{Pos}(0,0)$=6.03, $U_{Neg}(0,0)$=1.8.

In Fig. 1 we can see how the central heights of both solitons change during propagation. Note that the heights of the two fields initially decrease, but after a distance of about 70, fairly regular oscillations of both amplitudes remain in the system with a small relative amplitude of $3.4\cdot10^{-4}$ for U_{Pos} and 5 times less for U_{Neg}. Moreover, we can see that the amplitude of the *Neg* soliton changes rapidly at the very initial stage of propagation.

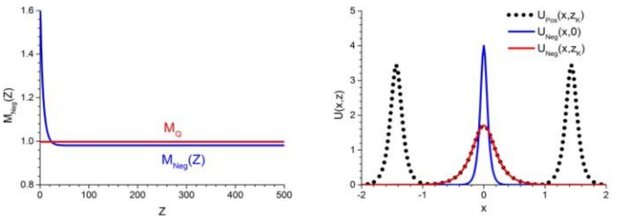

Fig. 2. The change of power of the propagating *Neg* field (left) and comparison of the initial and final profiles (right).

Analogous behaviour can be seen for the power $M_{Neg} = \int_{-\infty}^{\infty} |U_N(x,z)|^2 \, dx$ of the *Neg* soliton (left graph of Fig. 2). On the other hand, in the right graph one can observe that the profile of the *Neg* field and the central peak of the *Pos* field agree very well (the *Pos* field is scaled to obtain the same central heights). Adjusting *Sech* function to the final profile of both beams, we find η_d = 2.9686, which is 1% less than the assumed value η_P = 3 for the initial central peak. For the initial shapes of both considered solitons, we obtain Q = 0.7894, which gives the estimated value of power with a fairly good 1% accuracy (the red line). Using the obtained η_d, one can calculate $\gamma_N^{(1)}$ = 0.2874 and $\gamma_P^{(2)}$ = 0.0103, which gives the final height of the *Neg* soliton drawn by the red line in the right graph of Fig. 1 (accuracy 0.2% nevertheless is a significant change) and the final height of the *Pos* soliton drawn by the black line in the left graph (accuracy 0.01%).

Of course, the accuracy of the method increases for closer values of the central widths (closer η_P and η_N). But it is also important to have a clearly marked peak in the centre of the *Pos* soliton field distribution. In Fig. 3 we show the final fields for the initial central peak height equal 0.3 (the left graph) and 0.25 (the right graph) of the side peaks. All other parameters are the same. In the left graph one can hardly distinguish deformations (however both final fields lost their symmetry), but in the right graph we can see that propagation became unstable – the *Neg* field is no longer guided by *Pos* soliton.

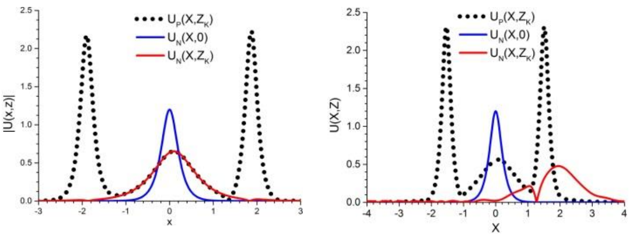

Fig. 3. Deformations of the final profiles of *Neg* solitons.

In conclusions, using the perturbation theory we have found solutions in the form of a pair of solitary beams with different orders, propagating in the focusing-defocusing medium. The proposed method gives very good quantitative results and calculated beams are stable at long distances. Similar multi-hump solitons have been reported recently, however, in a nonlocal nonlinear medium [13-15].

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