In this paper, we seek special soliton solutions of HNLS, namely exact traveling wave solutions ([5], Chapter 7). As it has been emphasized in [8], these solutions are some of the most fundamental objects of study in mathematical physics. Using a developed Jacobi elliptic function expansion method described in [8], we show that in the case of four-order dispersion these traveling wave solutions do not exist. Hence, for ultrashort pulses, it is not necessary to include orders of dispersion which are higher than three to obtain solutions of this type. As a byproduct, we regain the results obtained by other authors [8, 9] for the case when only the dispersion to the third order is taken into account.

Our paper is organized as follows. In Section 2, following [8] there is a short description of the developed Jacobi elliptic function expansion method. In Section 3, by using this method for the HNLS with the four-order dispersion, we show that traveling wave solutions do not exist in this case. We also regain dark and bright solitons obtained by other authors [8, 9]. The last section contains our conclusions.

2. Developed Jacobi elliptic function expansion

We consider a nonlinear partial differential equation in a general form

\[ N(F,F_t,F_{tt},F_{tx},F_{xx},F_{xt},F_{ttt},F_{xtt},F_{xxx},F_{xxt},F_{tttx},F_{ttttx}) = 0. \] (2.1)

We seek the traveling wave solutions of the form

\[ F = u(\xi)e^{\xi/(\omega - i k)}, \quad \xi = \omega t - \lambda x + \xi_0, \] (2.2)

where \( u(\xi) \) is a real function, \( \lambda \) is a constant parameter and \( k \) and \( \omega \) denote the wave number and frequency, respectively. Substituting (2.2) into (2.1) we obtain an ordinary differential equation

\[ N\left( u, \frac{du}{d\xi}, \frac{d^2u}{d\xi^2}, \frac{d^3u}{d\xi^3}, \ldots \right) = 0. \] (2.3)

We take the ansatz of the solution in the form of a finite series of Jacobi elliptic functions \( cn(\xi m) \) (or \( sn(\xi m) \)), i.e.

\[ u(\xi) = \sum_{j=0}^{N} A_j cn^{(j)}(\xi), \] (2.4)

in which \( A_j \) are constants which will be determined later and the highest degree of the function \( u \) is...
It follows from the properties of Jacobi elliptic functions that the highest degree of derivatives is taken as
\[ O(d^n u(\xi)/d\xi^n) = n. \]  
(2.5)

3. The higher-order propagation equation

As it has been emphasized in Sec. 1, a systematic derivation for light propagation equation in a nonlinear medium is given in [1, 2]. In the case of ultrashort light pulses (femtosecond pulses which have much potential for future technology), in comparison with the nonlinear Schrödinger equation (NLS), higher-order terms should be taken into account. For this reason, we consider the modified NLS equation in the form
\[ \frac{\partial E}{\partial z} = \left( a_1 \frac{\partial^2 E}{\partial t^2} + a_2 \frac{\partial^4 E}{\partial t^4} + a_3 \left[ \frac{\partial E}{\partial t} \right]^2 \right) + a_4 \frac{\partial^5 E}{\partial t^5}, \]  
(3.1)

where the real parameters \( a_i \) (\( i = 1, \ldots, 6 \)) have the following physical interpretations: \( a_1 \) corresponds to the group velocity dispersion (GVD), \( a_2 \) to the four-order dispersion (FOD), \( a_3 \) to self-phase modulation (SPM), \( a_4 \) to third-order dispersion (TOD), \( a_5 \) to self-steepening (SS) and \( a_6 \) to the self frequency shift (SFS) arising from stimulated Raman scattering (SRS). In order to find traveling wave solutions of Eq.(3.1), we use the developed Jacobi elliptic function expansion method described above. Firstly, we write the electric field in the form
\[ E(z,t) = u(\xi) \exp ikz + io_{\omega}(z - \omega t + z_0). \]  
(3.2)

Substituting (3.2) into (3.1) we obtain
\[ -iu_{\omega} + ik u = \kappa [c^2 u'' + co \omega u - 6c^2 c'o'^2 u + 4c' c'o' o^{'2} u] \]  
+ \[ a_1 (c^2 u'' - 2c' c'o' u + co' u')^2 + a_2 (c^2 u' - co u')^3, \]  
(3.3a)

Separating the real and imaginary parts of this equation leads to the following system of equations
\[ c^2 (\alpha_1 + 4c', c o \omega) u'' + (\lambda - 3c, c o^2 \omega + 2c, c o \omega - 4c, c o^3) u' + (3\alpha_4 + 2c, c o \omega) u^2 u' = 0, \]  
(3.4a)

\[ c^4 (\alpha_2 + 6c, c o^2 \omega - 3c^2 o^3 \omega) u'' + (k - \alpha_3 o^3 - 6c^2 c' \omega u + (\lambda \alpha_3 - \alpha_3 o \omega) u = 0. \]  
(3.4b)

Now using developed Jacobi elliptic function expansion method described in Sec. 2 we calculate the highest degree of derivative \( O(d^u u(\xi)/d\xi^u) = n + 4 \) and the degree of the nonlinear term \( O(\partial^u u(\xi)/\partial^{n+4}\xi) = 3n \). Equating these numbers leads to \( n = 2 \). It follows that the function \( u(\xi) \) can be obtained from the form
\[ u(\xi) = a_0 + a_2 cn(\xi) + a_3 cn^2(\xi). \]  
(3.5)

For the sake of simplicity, we propose that \( a_0 = a_2 = 0 \), then \( u(\xi) = a_3 cn(\xi) \). Substituting this expression into Eq. (3.4b) gives
\[ A_4 cn^4(\xi) + 2A_5 cn^4(\xi) + A_6 cn(\xi) = 0, \]  
where coefficients \( A_i \) contain different parameters involved in the problem. Equating the coefficient of the first term in (3.6) to zero leads to \( A_6 = 48A_4 c^3 a_4 m^2 + 24c^3 a_4 m^2 = 0 \). Because \( a_2, m, c \) should be different from zero, we have \( a_3 = 0 \). This means that if the term FOD is taken into account, the traveling wave solutions do not exist. We conclude that for the existence of solutions in this type, the orders of dispersion higher than three should not be taken into account. Then we can rewrite (3.4a) and (3.4b) in the form (with \( a_3 = 0 \))
\[ c^4 (\alpha_2 - 3c^2 c' \omega) u'' + (k - \alpha_3 o^3 + \alpha_3 o \omega) u + (\lambda \alpha_3 - \alpha_3 o \omega) u = 0. \]  
(3.7a)

Differentiating two sides of Eq. (3.7b) with respect to the \( \xi \) gives us
\[ u'''' + (k - \alpha_3 o^3 + \alpha_3 o \omega) u' + (\lambda \alpha_3 - \alpha_3 o \omega) u^2 u' = 0. \]  
(3.8)

Comparing (3.8) with (3.7a) leads to formulas for \( \omega \) and \( k \):
\[ \omega = [\alpha_3 (a_3 + a_6)] - [\alpha_3 (a_3 + a_6)] \]  
(3.9)

\[ k = \frac{1}{\alpha_4} (3 \alpha_3 - \alpha_3 o^2 - \alpha_3 c o \omega) - \alpha_3 o^2 + \omega a_4. \]  
(3.10)

Then Eqs. (3.7a), (3.7b) reduce to
\[ u'' + Au + Bu' = 0, \]  
(3.11)

where
\[ A = \frac{2\alpha_3 c o \omega - \lambda - 3\alpha_3 o^3}{2c^3 a_4}, \quad B = \frac{3a_3 + 2a_6}{3c^3 a_4}. \]  
(3.12)

Now, we use the formalism described in Sec.2 for the Eq.(3.11). Firstly, we calculate \( O(d^u u(\xi)/d\xi^u) = n + 2 \) and \( O(\partial^u u(\xi)/\partial^{n+2}\xi) = 3n \). Then \( n + 1 \) and we can write \( u(\xi) \) in the following form:
\[ u(\xi) = a_0 + a_2 cn(\xi). \]  
(3.13)

Substituting (3.13) into (3.11) and equating the coefficients of all powers of \( cn(\xi) \) to zero yields the values of unknown parameters \( a_0, a_1, c, \lambda \). We have performed this step by MAPLE and obtained:
\[ a_0 = 0, a_1 = \frac{6a_4 \text{cs}}{2a_2 + 3a_4 c}, \quad \lambda = -2m^2 c^3 a_4 - 2c^3 a_4 c^2 + 3c a_3 c^2. \]  
(3.14)

while \( c \) is an arbitrary constant and \( m \) is the modulus number of the Jacobi elliptic functions. Then the traveling wave solutions of the propagation equation (3.1) have the following form:

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\[ E(z,t) = \sqrt{\frac{6\alpha_4}{2\alpha_4 + 3\alpha_3}} \exp[i(kz - \omega t)] \]  
(3.15)

where the expressions for \( k \) and \( \omega \) are given by (3.9) and (3.10). When \( m \) tends to 1, we obtain a bright soliton solution

\[ E(z,t) = \sqrt{\frac{6\alpha_4}{2\alpha_4 + 3\alpha_3}} \exp[i(kz - \omega t)] \]

\[ \cdot \exp[i\left(-c^2 \alpha_4 - 2\alpha_3 \omega + 3\alpha_4 \phi^2\right)z + z_0] \]  
(3.16)

where the expressions for \( \phi \) are given by (3.9) and (3.10). When \( m \) tends to 1, we obtain a bright soliton solution

\[ E(z,t) = \sqrt{\frac{-6\alpha_4}{2\alpha_4 + 3\alpha_3}} \exp[i(kz - \omega t)] \]

\[ \cdot \exp[i\left(c^2 \alpha_4 - 2\alpha_3 \omega + 3\alpha_4 \phi^2\right)z + z_0] \]  
(3.17)

where the expressions for \( \phi \) are given by (3.9) and (3.10). When \( m \) tends to 1, we have a dark soliton solution in the following form:

\[ E(z,t) = \sqrt{\frac{-6\alpha_4}{2\alpha_4 + 3\alpha_3}} \exp[i\left(-c^2 \alpha_4 - 2\alpha_3 \omega + 3\alpha_4 \phi^2\right)z + z_0] \]  
(3.18)

4. Conclusions

In our paper we used a developed Jacoby elliptic function expansion method to find traveling wave solutions for HNLS which describes light propagation in fibers. We showed that we cannot have solutions of this type when we include higher orders of dispersion. As a byproduct, we obtained in a simple way bright and dark soliton solutions discovered before by other authors.

References