Electromagnetic wave polarization state evolution in weakly anisotropic and nonuniform media with dissipation

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Abstract—The change of the polarization state is analysed of an electromagnetic beam propagating in weakly anisotropic and smoothly inhomogeneous media with dissipation. On the basis of a quasi-isotropic approximation, which provides a consequent asymptotic solution of Maxwell's equation, a differential equation for the evolution of a four-component Stokes vector is derived. The obtained equation generalizes the previous results for nonadsorbing media and is written in terms of a dielectric tensor of birefringent media with dissipation. The formalism is illustrated by an example of magnetised plasma with dissipation due to electron collisions.

Traditional methods for the analysis of the polarization state of an electromagnetic wave passing through nonuniform birefringent media have been concentrated on the case of negligible wave absorption. In such a case the polarization state evolution is limited to a combination of polarization plane rotation (the Faraday effect in media with circular birefringence) and change in the ellipticity state (Cotton-Mouton effect in media with linear birefringence). To describe such evolution there are two main approaches. The first one deals with coupled wave equations for the components of an electromagnetic wave field. In stratified media it is Budden's method [1-2], whereas the modification of Budden's approach for an arbitrarily inhomogeneous medium is a quasi-isotropic approximation (QIA) of geometric optics [3-6]. The main disadvantage of QIA is the fact that for an electromagnetic wave it describes the evolution of amplitude and phase of each perpendicular component of an electric field whereas from polarimetric measurements information is obtained on the amplitude of both field components and only their phase difference. The second, alternative approach – the Stokes vector formalism (SVF) for the electromagnetic wave in stratified media, initiated in [7] and reviewed in [8], was applied for purposes of e.g. fiber and plasma polarimetry. In the case of a medium without dissipation, SVF – based on the reduced three-component Stokes vector $\mathbf{s}$ is completely sufficient.

The purpose of this work is to derive an evolution equation for the full four-component Stokes vector directly from the QIA. The obtained Stokes vector equation generalizes the previous results for refractionless and nonadsorbing media. The paper is organized as follows. At the beginning, basic QIA equations are presented. Next, equations for the four-component Stokes vector are derived from the QIA for a weakly anisotropic medium of a general type. Then, a general theory is applied for Stokes-vector evolution in weakly anisotropic collisional plasma, such ionosphere, laboratory or tokamak plasma with parameters typical for modern tokamak machines.

The theory of electromagnetic wave propagation in weakly anisotropic media was developed in [3] in the form of a quasi-isotropic approximation of the geometric optics method. A short outline of the QIA is presented in the books [4-5] and in the review paper [6]. The quasi-isotropic approximation of geometrical optics is efficient for describing the evolution of electromagnetic wave components in a 3D weakly inhomogeneous weakly anisotropic medium. The dielectric permittivity ε_{ij} of a weakly anisotropic medium could be split into two parts: a large isotropic component $\varepsilon_0\delta_{ij}$, where δ_{ij} is the unit vector, and an anisotropic component ν_{ij}, which is small as compared with the isotropic part ε_0: $\max|\nu_{ij}| \ll \varepsilon_0$:

$$\varepsilon_{ij} = \varepsilon_0\delta_{ij} + \nu_{ij}, \quad (1)$$

The quantity $\mu_a = \max|\nu_{ij}|/\varepsilon_0 \ll 1$ is an "anisotropic" small addition to the traditional small parameter of geometrical optics $\mu_{GO} = 1/\mathrm{kL} \ll 1$, where k is the wave number and L is the characteristic scale of an inhomogeneous medium. Thy QIA comes from the solution of Maxwell's equations by asymptotic expansion of the electromagnetic wave field $\mathbf{E}$ with respect to the combined small parameter $\mu = \max(\mu_{GO}, \mu_a)$. In the lowest order of the QIA, the monochromatic electromagnetic wave field $\mathbf{E}$ has the form of a transverse wave:

$$\mathbf{E} = E_x\boldsymbol{e}_x + E_y\boldsymbol{e}_y = A\left(\Gamma_x\boldsymbol{e}_x + \Gamma_y\boldsymbol{e}_y\right)\exp[ik\Psi]. \quad (2)$$

In the frame of the QIA theory a weak anisotropy influences the polarization vector $\mathbf{\Gamma} = \Gamma_x\boldsymbol{e}_x + \Gamma_y\boldsymbol{e}_y$ rather than the eikonal Ψ and the wave amplitude A: the latter obey the same equations as in the isotropic medium. The polarization vector components Γ_x and Γ_y obey the QIA

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coupled equations, which can be presented in the form [6]:

$$\begin{cases} \Gamma_x' = \frac{1}{2} ik\varepsilon_0^{-1/2}\left(\nu_{11}\Gamma_x + \nu_{12}\Gamma_y\right), \\ \Gamma_y' = \frac{1}{2} ik\varepsilon_0^{-1/2}\left(\nu_{21}\Gamma_x + \nu_{22}\Gamma_y\right), \end{cases} \quad (3)$$

where derivatives are calculated over dz – the elementary arc length of the ray.

The four-component Stokes vector $\boldsymbol{S}$ is defined by [9]:

$$\boldsymbol{S} = \begin{bmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{bmatrix} = \begin{bmatrix} \Gamma_x\Gamma_x^* + \Gamma_y\Gamma_y^* \\ \Gamma_x\Gamma_x^* - \Gamma_y\Gamma_y^* \\ 2\mathrm{Re}\left(\Gamma_x\Gamma_y^*\right) \\ 2\mathrm{Im}\left(\Gamma_x\Gamma_y^*\right) \end{bmatrix} = \begin{bmatrix} \Gamma_x\Gamma_x^* + \Gamma_y\Gamma_y^* \\ \Gamma_x\Gamma_x^* - \Gamma_y\Gamma_y^* \\ \Gamma_x\Gamma_y^* + \Gamma_x^*\Gamma_y \\ -i\left(\Gamma_x\Gamma_y^* - \Gamma_x^*\Gamma_y\right) \end{bmatrix}. \quad (4)$$

To obtain the evolution equation of the Stokes vector in a weakly anisotropic medium, the derivative of (4) has to be calculated with appropriate replacements of Γ_x' and Γ_y' from equation 3. For example, for component S_1 such a calculation has the form:

$$S_1' = \left(\Gamma_x\Gamma_x^* - \Gamma_y\Gamma_y^*\right)' = \Gamma_x'\Gamma_x^* + \Gamma_x\Gamma_x'^* - \Gamma_y{}'\Gamma_y^* - \Gamma_y\Gamma_y'^* =$$
$$\frac{1}{2}ik\varepsilon_0^{-\frac{1}{2}}\left(\nu_{11}\Gamma_x\Gamma_x^* + \nu_{12}\Gamma_y\Gamma_x^* - \nu_{11}^*\Gamma_x\Gamma_x^* - \nu_{12}^*\Gamma_x\Gamma_y^*\right)$$
$$-\frac{1}{2}ik\varepsilon_0^{-\frac{1}{2}}\left(\nu_{21}\Gamma_x\Gamma_y^* + \nu_{22}\Gamma_y\Gamma_y^* - \nu_{21}^*\Gamma_y\Gamma_x^* - \nu_{22}^*\Gamma_y\Gamma_y^*\right) =$$
$$\frac{1}{2}ik\varepsilon_0^{-\frac{1}{2}}\left((\nu_{11} - \nu_{11}^*)\Gamma_x\Gamma_x^* - (\nu_{22} - \nu_{22}^*)\Gamma_y\Gamma_y^* - \right.$$
$$\nu 21+\nu 12*\Gamma x\Gamma y*+\nu 12+\nu 21*\Gamma y\Gamma x*. \quad (5)$$

For further calculations it is convenient to introduce four-component complex vector $\boldsymbol{G}$:

$$\boldsymbol{G} = \begin{bmatrix} G_0 \\ G_1 \\ G_2 \\ G_3 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}ik\varepsilon_0^{-\frac{1}{2}}(\nu_{11} + \nu_{22}) \\ \frac{1}{2}ik\varepsilon_0^{-\frac{1}{2}}(\nu_{11} - \nu_{22}) \\ \frac{1}{2}ik\varepsilon_0^{-\frac{1}{2}}(\nu_{12} + \nu_{21}) \\ -\frac{1}{2}ik\varepsilon_0^{-\frac{1}{2}}(\nu_{12} + \nu_{21}) \end{bmatrix}. \quad (6)$$

Using Eqs. (4) and (6), it is possible to rewrite (5) in the form:

$$S_1' = -\mathrm{Im}(G_1)S_0 + \mathrm{Im}(G_0)S_1 - \mathrm{Re}(G_3)S_2 + \mathrm{Re}(G_2)S_3. \quad (7)$$

The other $\boldsymbol{S}'$ vector components are calculated in a similar way. Finally, the equation for the evolution of the Stokes vector can be written in the matrix form:

$$\mathbf{S}' = \mathbf{M} \cdot \mathbf{S} =$$
$$\begin{bmatrix} \mathrm{Im}(G_0) & -\mathrm{Im}(G_1) & -\mathrm{Im}(G_2) & -\mathrm{Im}(G_3) \\ -\mathrm{Im}(G_1) & \mathrm{Im}(G_0) & -\mathrm{Re}(G_3) & \mathrm{Re}(G_2) \\ -\mathrm{Im}(G_2) & \mathrm{Re}(G_3) & \mathrm{Im}(G_0) & -\mathrm{Re}(G_1) \\ -\mathrm{Im}(G_3) & -\mathrm{Re}(G_2) & \mathrm{Re}(G_1) & \mathrm{Im}(G_0) \end{bmatrix} \cdot \mathbf{S}, \quad (7)$$

where $\mathbf{M}$ is the Mueller matrix for a weakly anisotropic inhomogeneous medium and can be presented as a sum of three terms: $\mathbf{M} = \mathbf{M_a} + \mathbf{M_d} + \mathbf{M_b}$. The first one, the *attenuation* component,

$$\mathbf{M_a} = \begin{bmatrix} \mathrm{Im}(G_0) & 0 & 0 & 0 \\ 0 & \mathrm{Im}(G_0) & 0 & 0 \\ 0 & 0 & \mathrm{Im}(G_0) & 0 \\ 0 & 0 & 0 & \mathrm{Im}(G_0) \end{bmatrix}, \quad (8)$$

describes isotropic attenuation common for all components of the Stokes vector. The second one, the *dichroic* term:

$$\mathbf{M_d} = \begin{bmatrix} 0 & -\mathrm{Im}(G_1) & -\mathrm{Im}(G_2) & -\mathrm{Im}(G_3) \\ -\mathrm{Im}(G_1) & 0 & 0 & 0 \\ -\mathrm{Im}(G_2) & 0 & 0 & 0 \\ -\mathrm{Im}(G_3) & 0 & 0 & 0 \end{bmatrix}, \quad (9)$$

corresponds to the attenuation responsible for dichroism, that is, for selective attenuation of normal modes. The last term,

$$\mathbf{M_b} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\mathrm{Re}(G_3) & \mathrm{Re}(G_2) \\ 0 & \mathrm{Re}(G_3) & 0 & -\mathrm{Re}(G_1) \\ 0 & -\mathrm{Re}(G_2) & \mathrm{Re}(G_1) & 0 \end{bmatrix}, \quad (10)$$

describes *birefringence*. Therefore real and imaginary parts of vector $\boldsymbol{G}$ correspond to hermitian and antihermitian parts of a permittivity tensor and are involving, respectively, isotropic attenuation or dichroism and birefringence. Please note that the quantity $\mathrm{Re}(G_0)$ does not appear in (7), which means that the total phase of the wave is lost in the Stokes vector evolution calculation. Only the phase difference between polarization modes can be retrieved from $\mathrm{Re}(G_i)$, where $\mathrm{i} = 1,2,3$.

As an example, we will apply the obtained equations to the analysis of the Stokes vector evolution in nonrelativistic plasma in external magnetic field **B** with wave absorption due only to electron collisions (described by constant electron collision frequency ν_c) and without a kinetic effect, such as occurs for ionosphere, laboratory or tokamak plasma. For a sufficient high electromagnetic wave frequency ω such plasma is weakly anisotropic and the components of the permittivity tenor transverse to the ray propagating in the z direction have the form [10]:

$$\begin{cases} \varepsilon_{xx} = 1 - \frac{v\left((1+is)^2 - \bar{u}B_x^2\right)}{(1+is)\left((1+is)^2-u\right)} \\ \varepsilon_{xy} = \frac{v\left(i(1+is)\sqrt{\bar{u}}B_z + \bar{u}B_xB_y\right)}{(1+is)\left((1+is)^2-u\right)} \\ \varepsilon_{yx} = \frac{v\left(-i(1+is)\sqrt{\bar{u}}B_z + \bar{u}B_xB_y\right)}{(1+is)\left((1+is)^2-u\right)} \\ \varepsilon_{yy} = 1 - \frac{v\left((1+is)^2 - \bar{u}B_y^2\right)}{(1+is)\left((1+is)^2-u\right)} \end{cases} \quad (11)$$

where $\bar{u} = (e/m_e c\omega)^2$, $u = (\omega_c/\omega)^2 = (eB/m_e c\omega)^2$, $v = \left(\omega_p/\omega\right)^2 = 4\pi e^2 N_e/m_e\omega^2$ and $s = \nu_c/\omega$ are the standard parameters of plasma with density N_e. Weak anisotropy requires $u \ll 1$ or $v \ll 1$. Note that only one of the parameters u or v has to be small, while the other one can be comparable with unity. Assuming $\varepsilon_0 = 1 - v$ as the isotropic part of the permittivity tensor (plasma without magnetic field and absorption), so from (1) $\nu_{ij} = \varepsilon_{ij} - (1 - v)\delta_{ij}$, and weak dissipation in the plasma ($s \ll 1$), the components of the Mueller matrix **M** from (6) and (11) in the first approximation of s are:

$$\mathrm{Im}(G_0) = -\tfrac{1}{2}k\varepsilon_0^{-1/2}\frac{v\left[-(3-u)\bar{u}\left(B_x^2+B_y^2\right)+2(1+u)\right]}{(1-u)^2}s$$

$$\mathrm{Im}(G_1) = -\tfrac{1}{2}k\varepsilon_0^{-\frac{1}{2}}v\bar{u}\left(B_x^2-B_y^2\right)\frac{3-u}{(1-u)^2}s = -\Omega_1\frac{3-u}{1-u}s$$

$$\mathrm{Im}(G_2) = -\tfrac{1}{2}k\varepsilon_0^{-\frac{1}{2}}v\bar{u}2B_xB_y\frac{3-u}{(1-u)^2}s = -\Omega_2\frac{3-u}{1-u}s$$

$$\mathrm{Im}(G_3) = -\tfrac{1}{2}k\varepsilon_0^{-\frac{1}{2}}4v\sqrt{\bar{u}}B_z\frac{1}{(1-u)^2}s = -\Omega_3\frac{2}{1-u}s$$

It is worth noting that in such a medium both isotropic and anisotropic attenuation are proportional to the collision frequency ν_c and inversely proportional to the electromagnetic wave frequency ω.

$$\mathrm{Re}(G_1) = \tfrac{1}{2}k\varepsilon_0^{-\frac{1}{2}}v\bar{u}\left(B_x^2-B_y^2\right)\frac{1}{1-u} = \Omega_1 \sim 2.45\cdot10^{-11}\lambda^3\left(B_x^2-B_y^2\right)N_e$$

$$\mathrm{Re}(G_2) = \tfrac{1}{2}k\varepsilon_0^{-\frac{1}{2}}v\bar{u}2B_xB_y\frac{1}{1-u} = \Omega_2 \sim 2.45\cdot10^{-11}\lambda^3\left(2B_xB_y\right)$$

$$\mathrm{Re}(G_3) = \tfrac{1}{2}k\varepsilon_0^{-\frac{1}{2}}2v\sqrt{\bar{u}}B_z\frac{1}{1-u} = \Omega_3 \sim 5.26\cdot10^{-13}\lambda^2B_z$$

where coefficients Ω_i, traditionally used in plasma polarimetry [8], correspond to the Cotton-Mouton effects (Ω_1 and Ω_2) and to the Faraday phenomenon (Ω_3) [11,12]. In a nondissipative medium, $\mathrm{Im}(G_i) = 0$ and (7) reduce to the standard Stokes vector precession equation [8]:

$$\mathbf{s}' = \mathbf{\Omega} \times \mathbf{s}. \quad (12)$$

To summarize, we have derived an evolution equation for the full four-component Stokes vector of the electromagnetic wave propagating in a weakly anisotropic, smoothly inhomogeneous medium, with no negligible absorption, as a function of permittivity tensor components. As a starting point we used the equations of a quasi-isotropic approximation (QIA), which follow in a consequent asymptotic way from Maxwell's equations. Obtained equation 7 allows to investigate the polarization state evolution in any medium with a sufficiently small combined parameter μ, like different types of plasma (ionospheric, astrophysical, laboratory or tokamak plasma) or fibers.

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